INVESTIGATION OF THE STABILITY CHARACTERISTICS OF A COMPRESSIBLE BOUNDARY LAYER ON A FLAT PLATE AT HIGH MACH NUMBER

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This paper investigates the stability of the boundary layer on a flat plate washed by helium at high Mach number (M_∞ = 8-25). To determine the profiles of velocity and temperature of the unperturbed flow one must take into account the interaction of the boundary layer with the external flow. To solve this problem of the linear theory of stability of compressible flows we have constructed a pseudospectral method with which, using a comparatively small number of basic functions to model an approximate solution, we can calculate stability characteristics over a wide range of Reynolds and Mach numbers.

We have compared the stability characteristics obtained in one case for profiles computed with interaction and for another case for profiles of the Blasius similarity solution. We found that for $M_\infty \approx 20$ allowing for interaction leads to an increase of the critical Reynolds number R_{xc} by more than a factor of two.

In spite of an increasing interest in stability of hypersonic boundary layers [1-7] the very high Mach number region ($M_\infty > 10$) has evidently not been investigated adequately. The basic reason is that most papers study boundary layer stability in air, considered as a perfect gas. The range of M_∞, for which air can be so considered is very limited, both for natural flight and for wind tunnels, which means that one should plan to study the influence of chemical reactions in the boundary layer, and also the influence of surface catalyticity on boundary layer stability. A number of papers [8-10] have made progress in this direction, but the solution of the problem is far from complete.

On the other hand, in helium tunnels one can achieve very high values of M_∞ and R_x without condensation. One would expect that the characteristic features of purely hydrodynamic type observed in a helium tunnel at high M_∞ would appear also in air, superimposed on features associated with chemical reactions. The use of helium tunnels to study the development of instability and transition in the hypersonic boundary layer gives a practical basis for investigating boundary layer stability in a perfect gas for $M_\infty > 10$.

In this paper we investigate linear stability of the boundary layer on a flat plate located at zero angle of attack in hypersonic helium flow. We examine only distances from the leading edge for which the boundary layer concept can be applied accurately.

At high M_∞ the main difference of the boundary layer from the known similarity solution at constant pressure comes from interaction of the boundary layer with the external flow [11]. The influence of the entropy layer and the presence of a density discontinuity in the external flow are not considered. This approximation limits our study to the regime of weak and moderate interaction ($\chi \le 1$, $\chi = M_\infty^3 c_\infty^{1/2}/R_x^{1/2}$, $c_\infty = \rho_w\mu_w/\rho_\infty\mu_\infty$, $R_x = \rho_\infty u_\infty x/\mu_\infty$). However, one can expect that some qualitative results carry over to the region of large values of the parameter χ.

Following the Blasius and Dorodnitsyn–Lees transformations

$$z = \frac{1}{M_\infty^2}\int_0^x \frac{\rho_w u_e}{\mu_w}\,dx, \quad \eta = \frac{1}{M_\infty\sqrt{2z}}\frac{u_e}{\mu_w}\int_0^y \rho\,dy,$$
$$\partial f/\partial\eta = u/u_e, \quad g = H/H_e$$

the boundary layer equations can be written in the form [11]

$$\begin{gathered}(Nf'')' + ff'' - \beta(g - f'^2) = 2z\left(f'\frac{\partial f'}{\partial z} - f''\frac{\partial f}{\partial z}\right), \\ \left[\frac{N}{\sigma}g' + \alpha N\left(1 - \frac{1}{\sigma}\right)f'f''\right]' + fg' = 2z\left(f'\frac{\partial g}{\partial z} - g'\frac{\partial f}{\partial z}\right),\end{gathered} \tag{1}$$

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$$\beta = \frac{\gamma - 1}{\gamma}\frac{z}{p}\frac{dp}{dz}(1 + \varepsilon_e), \quad \alpha = \frac{2}{(1 + \varepsilon_e)},$$

$$N = \left[\frac{(1 + \varepsilon_e) g_w}{(1 + \varepsilon_e) g - f'^2}\right]^{1-\omega}, \quad \gamma = \frac{5}{3}, \quad \sigma = \frac{2}{3}, \quad \omega = 0{,}647,$$

$$\varepsilon_e = \frac{2}{(\gamma - 1) M_e^2} = \varepsilon_\infty \Big/ \left[(1 + \varepsilon_\infty)\left(\frac{p_\infty}{p}\right)^{\frac{\gamma-1}{\gamma}} - \varepsilon_\infty\right], \quad \frac{u_e}{u_\infty} = \left(\frac{1 + \varepsilon_\infty}{1 + \varepsilon_e}\right)^{1/2},$$

$$\delta^* = \frac{\gamma - 1}{\gamma}\frac{\sqrt{2z}}{p} u_\infty F \mu_w M_\infty, \quad F = \frac{u_e}{u_\infty}\int_0^\infty [(1 + \varepsilon_e) g - f'^2 - \varepsilon_e f']\, d\eta,$$

$$f(z, 0) = f'(z, 0) = 0, \quad g(z, 0) = g_w = \text{const},$$

$$f'(z, \infty) = g(z, \infty) = 1.$$

Here x and y are the longitudinal and normal coordinates; u is the velocity; H is the total enthalpy; ρ is the density; p is the pressure; μ is the viscosity; δ^* is the displacement thickness; the primes denote differentiation with respect to η; the subscript ∞ refers to values in the unperturbed flow, e at the outer edge of the boundary layer, and w at the wall. We assume that for helium $\gamma = c_p/c_V = 5/3$, and Prandtl number is $\sigma = 2$, $\mu/\mu_\infty = (T/T_\infty)^\omega$, and $\omega = 0.647$.

The induced pressure gradient can be determined approximately from the tangent wedge formula [11-13]

$$\frac{p}{p_\infty} = 1 + \gamma k^2\left(\frac{\gamma + 1}{4} + \sqrt{\left(\frac{\gamma + 1}{4}\right)^2 + \frac{1}{k^2}}\right) = P(k), \quad k = M_\infty \frac{d\delta^*}{dx}. \tag{2}$$

We compute the pressure distribution in the limit $M_\infty \to \infty$, $z = O(1)$, $\varepsilon_e \to 0$, $\varepsilon_\infty \to 0$, when the solution of the problem of Eqs. (1) and (2) depends only on the coordinate z and the temperature factor g_w (the moderate hypersonic interaction limit). For qualitative investigations of the influence of interaction on stability it is appropriate to restrict ourselves to an approximate method based on local similarity of solutions of Eq. (1). The essence of the method is that the convective terms, equal to zero for $z \to 0$ and $z \to \infty$, are put equal to zero for the entire interval $0 < z < \infty$. Then the variable z enters Eqs. (1) and (2) only via the parameter β, and from the solution of this problem one can find the dependence $F(\beta)$ in the $(\gamma - 1)/\gamma < \beta < 0$ (from strong to weak interaction). Taking into acount the relation $\beta = \frac{\gamma - 1}{\gamma}\frac{z}{p}\frac{dp}{dz}$ and the interaction equations (2) we obtain a boundary problem for an ordinary differential equation of second order determining $\beta(\xi)$, $\xi = \ln z$, $\xi \in (-\infty, \infty)$:

$$\frac{dF}{d\beta}\frac{d^2\beta}{d\xi^2} - \frac{\gamma}{\gamma - 1}\left(\beta\frac{dF}{d\beta} + F\right)\frac{d\beta}{d\xi} + \frac{d^2F}{d\beta^2}\left(\frac{d\beta}{d\xi}\right)^2 - \frac{1}{4}F +$$

$$+ \frac{1}{2}\frac{\gamma}{\gamma - 1}\beta F = \frac{e^{\xi/2}}{\sqrt{2}} g_w \frac{\gamma}{\gamma - 1}\beta P(k) \Big/ \frac{dP}{dk},$$

$$k = e^{-\xi/2}\frac{\sqrt{2}}{g_w}\left(\frac{dF}{d\beta}\frac{d\beta}{d\xi} + \frac{1}{2}F - \frac{\gamma}{\gamma - 1}\beta F\right),$$

$$\beta(-\infty) = -(\gamma - 1)/\gamma, \ \beta(\infty) = 0.$$

From the solution of this problem we can obtain the function

$$\frac{p}{p_\infty}(z), \quad U = \frac{u}{u_e} = f'(\beta, \eta), \quad \frac{T}{T_w} = \frac{g - f'^2}{g_w}$$

(T is the temperature) and determine the link between z and the hypersonic similarity parameter $\chi = M_\infty^3 c_\infty^{1/2}/R_x^{1/2}$:

$$\chi = \frac{2}{g_w(\gamma - 1)} \Big/ \left[\int_0^z \frac{p_\infty}{p} dz\right]^{1/2}, \quad c_\infty = \frac{\rho_w \mu_w}{\rho_\infty \mu_\infty} = \left[\frac{2}{(\gamma - 1) M_\infty^2 g_w}\right]^{1-\omega},$$

$$R_x = M_\infty^6 c_\infty \frac{g_w^2 (\gamma - 1)^2}{4}\int_0^z \frac{p_\infty}{p} dz.$$

The profiles of velocity and temperature obtained in the hypersonic approximation ($\varepsilon_e = \varepsilon_\infty = 0$) are not uniformly appropriate as $M_\infty \to \infty$. Near the external boundary there is a thin region where attenuation of velocity perturbations varies from algebraic to exponential, and the temperatures goes to its nonzero value in the external flow. In this region the second derivatives of velocity and temperature with respect to the normal coordinate increase without bound with increase of M_∞.

Therefore, to find the profiles we chose the following procedure. The profiles of velocity and temperature are determined beforehand in the local similarity approximation, i.e., as a solution of Eq. (1) without the right-hand sides. However, then there remain the small terms ε_∞, ε_e in the equations. The pressure distribution, necessary for finding ε_e, is found by solving the problem in the hypersonic approximation ($\varepsilon_\infty = \varepsilon_e = 0$). For small enough ε_e these additions do not noticeably influence the pressure distribution obtained in the hypersonic approximation, and lead to the correct behavior of the profiles near the outer edge of the boundary layer. The temperature and velocity profiles thus found, and also their derivatives, are used subsequently to compute the stability characteristics.

We determine the stability characteristics in the locally homogenous flow approximation. We represent small perturbations of velocity, pressure, density, and temperature in the form of a Tollmien–Schlichting wave:

$$\begin{gathered}\mathbf{F}(x, y, z, t) = \{u', v', w', p', \rho', T'\} = \\ = \{u_1(y), \alpha v_1(y), w_1(y), p_1(y), \rho_1(y), T_1(y)\} \exp[i(\alpha x + \beta z - \alpha c t)],\end{gathered} \tag{3}$$

where v', w' are perturbations of the normal and transverse velocity components. The functions are referenced to their values in the incident stream: length to $L = x/R_x^{1/2}$, and time to L/u_∞. By linearizing the Navier–Stokes equations and substituting Eq. (3) we come to the system of equations of [14]. If the wall is fixed, impermeable, and has high thermal conductivity the boundary conditions for $y = 0$ have the form $u_1(0) = v_1(0) = w_1(0) = T_1(0) = 0$. For Tollmien–Schlichting waves moving subsonically relative to the external flow this becomes the condition for attenuation of perturbations as $y \to \infty$.

We now investigate the temporal stability for which the parameters of the problem are $R = (u_\infty \rho_\infty L/\mu_\infty)^{1/2}$, M_∞, the wave number α and the angle of slope of the wave vector $\psi = \arctan(\beta/\alpha)$. We require to find the eigenvalues c and the corresponding eigenfunctions. In the computations it is convenient to convert in the equations to the Dorodnitsyn–Lees variable η, and thus avoid large gradients in the vicinity of the outer edge of the boundary layer. the coefficient of volume viscosity is assumed to be zero.

The problem of boundary layer stability in compressible flows has been investigated in many papers. Mack [14] has made a large contribution. The contemporary state of the matter is explained in [15, 16]. In our work we propose an efficient new method of solving the problem, based on spectral representation of the solution.

The problem can be formulated in the form

$$\begin{gathered}c\mathbf{X} = A\mathbf{X},\ \mathbf{X} = \{X_n\} \quad (n = 0, 1, \ldots, 5), \\ X_n(\eta = 0) = 0 \quad (n = 1, 2, \ldots, 4), \\ X_n(\infty) = 0 \quad (n = 1, 2, \ldots, 5),\end{gathered} \tag{4}$$

where $\mathbf{X} = \{X_n\} = \{\tilde{p}|_{\eta=0}, \tilde{u}(\eta), \tilde{v}(\eta), \tilde{w}(\eta), \tilde{T}(\eta), \tilde{p}(\eta) - (1/T(\eta))T(\eta)\}$; $\tilde{u} = u_1 + w_1 \tan\psi$; $\tilde{v} = v_1$; $\tilde{w} = w_1 - u_1 \tan\psi$; $\tilde{T} = T_1$; $\tilde{p} = p_1$; $A = A(R, M_\infty, \alpha, \psi)$ is a known matrix, including an operator to differentiate with respect to η.

The problem of Eq. (4) is solved numerically using a modified pseudospectral method [17]. Using the substitution of variable $\eta = a(1 + z)/(1 - z)$ we map the region $\eta \in [0, \infty)$ into the interval $z \in [-1, 1)$, and here $d/d\eta = \kappa(z)(d/dz)$, $\kappa(z) = (1 - z)/2a$. The approximate solution of Eq. (4) is sought in the form

$$\begin{gathered}X_n(z) = (1 - z^2) \sum_{q=0}^{Q-1} \widehat{X}_{n,q} T_q(z), \quad n = 1, 2, \ldots, 4, \\ X_5(z) = \frac{1}{2}(1 - z) X_0 + (1 - z^2) \sum_{q=0}^{Q-1} \widehat{X}_{5,q} T_q(z)\end{gathered} \tag{5}$$

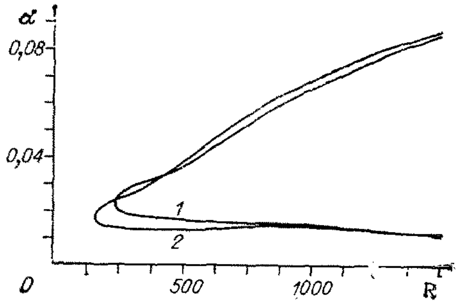

Fig. 1

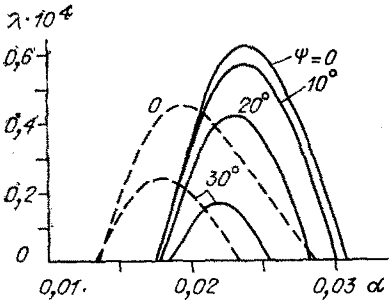

Fig. 2

TABLE 1

Q	$M_\infty = 18$, $R = 1000$, $\alpha = 0{,}04$	$M_\infty = 25$, $R = 1700$, $\alpha = 0{,}02$
	c_{max}	
31	0,988152 + i0,001846	0,99288 + i0,00233
41	0,988152 + i0,001980	0,992773 + i0,002068
51	0,98815123 + i0,00198119	0,9927754 + i0,0020720
71	0,98815122 + i0,00198118	0,9927748 + i0,0020721

[$T_q(z) = \cos(q \arccos z)$ are Chebyshev polynomials]. It follows from the form of Eq. (5) that the boundary conditions are satisfied.

Each equation of the system (4) is described at the collocation nodes $z_i = \cos(\pi i/(Q + 1))$, $i = 1, 2, \ldots, Q$; in addition the last equation ($n = 5$) is written at the point $z = -1$. As a result we obtain the problem of eigenvalues and eigenvectors for a complex matrix of dimension $(5Q + 1) \times (5Q + 1)$.

The accuracy of computing the approximate solution was evaluated for air and helium for the different parameters of the problem M_∞, R, α, ψ. In all the computations acceptable accuracy was achieved for $Q \geq 30$. Table 1 shows eigenvalues of the velocity c_{max} (with maximum imaginary part) found for helium at high M_∞ ($g_w = 0.6$, $\psi = 0$, $\sigma = 2/3$).

To evaluate the influence of interaction effects at different parameters of the problem we made two series of computations: In the first case (INT) in the stability problem we used velocity and temperature profiles obtained by the above method, i.e., allowing for interaction of the boundary layer with the external flow; in the second case (BLS) we used the profiles of the Blasius similarity solution ($\beta = 0$).

Figure 1 shows neutral stability curves computed for helium at $M_\infty = 18$, $g_w = 0.6$, $\psi = 0$; curve 1 corresponds to the case INT, and curve 2 to BLS. As one would expect the largest changes occur for small R.

To allow for interaction leads to an increase in the critical Reynolds number from $R_c^{BLS} \simeq 162$ to $R_c^{INT} \simeq 235$, corresponding to an increase of $R_{x,c} = R$ by more than a factor of two. In addition, there is displacement of the nose of the neutral curve toward the large wave number region ($\alpha_c^{BLS} \simeq 0.018$, $\alpha_c^{INT} \simeq 0.0229$). With increase of R the neutral curves come together.

Figure 2 shows the dependence of the increment of the perturbations $\lambda = \alpha \operatorname{Im}(c)$ on α for $R = 350$, $M_\infty = 18$, $g_w = 0.6$ for various values of the slope angle (the solid curves correspond to the case INT, and the broken curves to BLS). It can be seen that for the flow parameters considered the maximum instability occurs for two-dimensional perturbations ($\psi = 0$). It is interesting that for a supercritical Reynolds number to allow for interaction in the two-dimensional case ($\psi = 0$) leads to destabilization ($\lambda_{max}^{INT} > \lambda_{max}^{BLS}$, $\lambda_{max} = \max\limits_{\alpha} \lambda$), and in the three-dimensional case for $\psi = 30°$ it leads to stabilization.

An interesting question is the influence of the temperature factor on flow stability in the boundary layer for high M_∞. The idea was put forward in [2] that in the hypersonic boundary layer with increase of M_∞ the influence of the temperature factor and the pressure gradient on the stability characteristics diminishes. This is based on removal of the critical layer still further from the wall to the outer edge of the boundary layer.

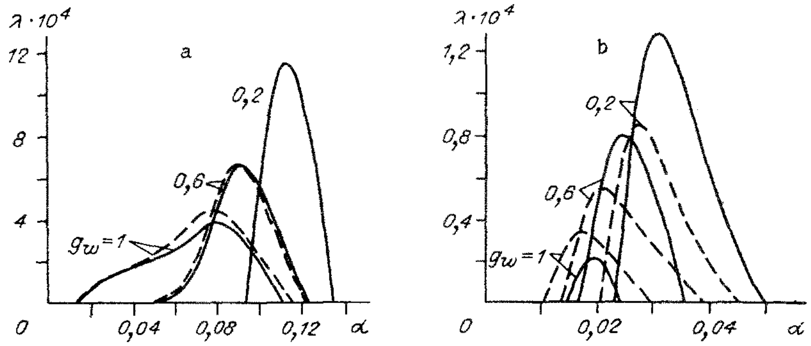

Fig. 3

Figure 3 shows, for M_∞ = 8 and 18 (a and b) the dependence of the increment of perturbation growth of λ on α for R = 500, ψ = 0 for various values of g_w (the solid curves are for INT, and the broken curves for BLS). It can be seen that a decrease of g_w leads to noticeable flow destabilization. If interaction is not accounted for with growth of M_∞ from 8 to 18 the influence of g_w varies only a little. However, allowing for interaction leads to the situation where in hypersonic regimes destabilization from decrease of g_w even increases, as proposed in [2]. Evidently, the temperature factor and the pressure gradient appreciably influence the intensity of vorticity in the vicinity of the outer edge of the boundary layer, where the critical layer is located for high M_∞.

In conclusion, we note that in this investigation we did not account for the influence of the entropy layer nor the shock wave. It is entirely probable that consideration of these effects for very high M_∞ can lead to qualitatively new variations in the behavior of the boundary layer stability characteristics.

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